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Abstract: Adjoint Functors and an Example in Logic.
Adjoint functors are pairs of functors which are in a particular sense inverses of one another. Using the concept, we are able to find structural relationships in Mathematics which were not before seen. One such example is the relationship between implication and conjunction in logic.

1. Introduction

Category theory has sought to provide deeper insight into mathematical structure. Adjoint functors have been useful in showing some profound relationships between previously unconnected notions.

Much pioneering work in Category theory including adjunction was done by Saunders MacLane and his students and the field is relatively young. Steven Awodey, one of MacLane's students once made the claim that, "... adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics." (3)

We will first explore some preliminary concepts in category theory leading up to adjunction, to provide the necessary insight and motivation for the concept. After describing the concept of adjoint functors, we will provide an example of one such pair.

2. Categories

A category **C** can be described by an ordered 6-tuple <C_0, C_1, d, c, m , e>

- C_0 is a class of objects
- C_1 is a class of morphisms (or maps or arrows)
- d and c map C_1 onto C_0, to provide a domain and codomain for each morphism

The domain and codomain, d(f) and c(f) respectively, of each morphism f are uniquely determined. For d(f) = A and c(f) = B we write $f: A \to B$.

- m is a binary operation on C_1 (in a naïve sense, because it can only act on composable pairs) (composition)

  Any $f: A \to B$ and $g: B \to C$ can be composed m(f, g) which we will abbreviate $gf: A \to C$ in C_1.

  m is associative.

- e maps every object A to a morphism $e(A) = 1_A: A \to A$ (i.e. a morphism 1_A with $c(1_A) = A = d(1_A)$) and for an

$f: B \to A$ and $g: A \to D$,

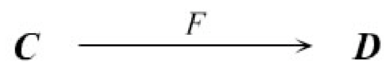
$$1_A f = f \qquad\qquad g 1_A = g$$

We call such a morphism 1_A the identity for A.

We said that C_0 here was a *class* of objects, and C_1 a *class* of morphisms. This allows C_0 to be for example, the class of all sets.

If C_0 is a *set*, we say **C** is a *small category.*

3. Functors



A functor F: **C** → **D** is a map which

- assigns to each object C in **C** an object F(C) in **D**

- assigns to each morphisms f:A → B in **C** a morphism

F(f):F(A) → F(B) in **D**

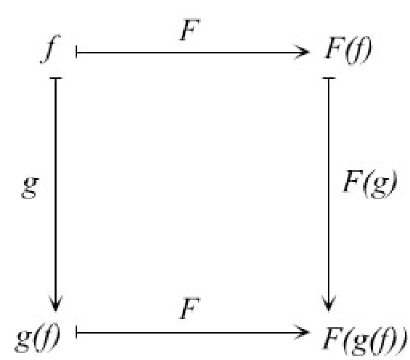
such the following conditions hold

- Functors must preserve identity morphisms.

When we apply a functor F: **C** → **D** to the identity morphism 1_X: X → X of X in **C**, we arrive at a morphism $F(1_X)$: F(X) → F(X). We require that this is the identity morphism for F(X) in **D**.

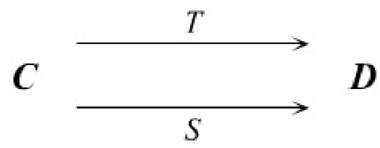
i.e. $1_{F(X)} = F(1_X)$

- We also require the following diagram to be commutative



That is to require: F(g(f)) = F(g)F(f)

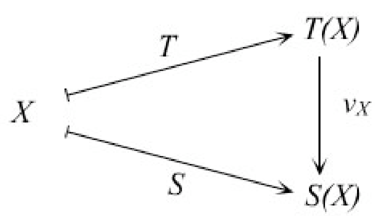
4. Natural Transformations



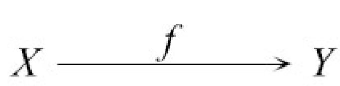
A natural transformation v: T → S (a map between functors) is a function which associates with every object X in C, a morphism in D

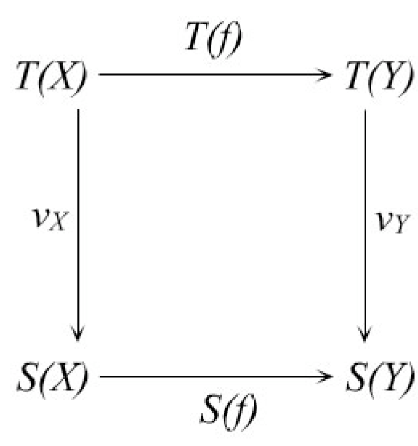
$$v_X : T(X) \rightarrow S(X)$$

As show in the diagram below, effect of T on X is changed into that of S on X.



The following diagram must commute for a natural transformation v.





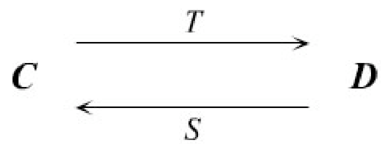
that is to say $v_Y(T(f)) = S(f)(v_X)$

Consider a category **P** with objects being functors from **C** to **D** (two fixed categories) and morphisms being natural transformations between these functors.

If $v:T \to S$, is an invertible morphism in **P**, then we call it a **natural isomorphism**.

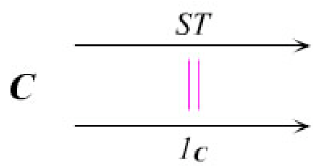
The morphisms $v_X:T(X) \to S(X)$ of such a v are isomorphisms for every X in **C**.

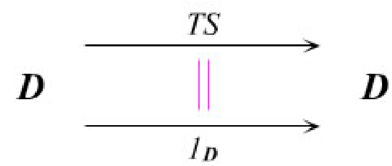
5. Isomorphic and Equivalent Categories



- **C** and **D** are **isomorphic** to each other if there exist functors T and S as above such that

$$ST = 1_{\mathbf{C}} \qquad \text{and} \qquad 1_{\mathbf{D}} = TS$$

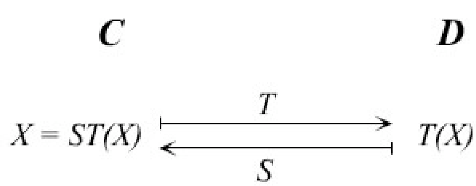




($1_{\mathbf{C}}$ is identity functor for **C**)

This would mean for an object X and morphism f in **C**, and for an object Y and morphism g in **D**

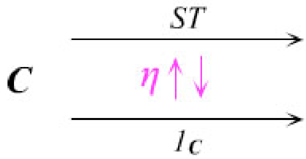
$$ST(X) = X \qquad \text{and} \qquad TS(Y) = Y$$
$$ST(f) = f \qquad \text{and} \qquad TS(g) = g$$

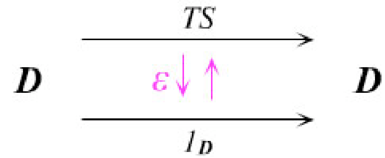


We say S and T are inverses if they satisfy these conditions.

- **C** and **D** are **equivalent** to each other if there exist functors T and S as above and natural isomorphisms η and ε such that

$$\eta{:}1_{\mathbf{C}} \rightarrow ST \qquad \text{and} \qquad \varepsilon{:}TS \rightarrow 1_{\mathbf{D}}$$



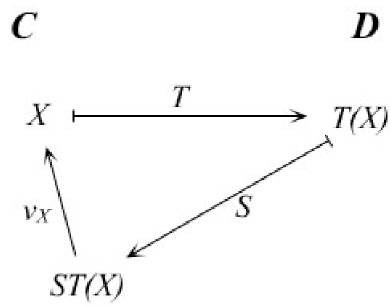


since these are isomorphisms, we also have

$$ST \approx 1_{\mathbf{C}} \qquad \text{and} \qquad 1_{\mathbf{D}} \approx TS$$

In particular, this would mean for an X in C and Y in D

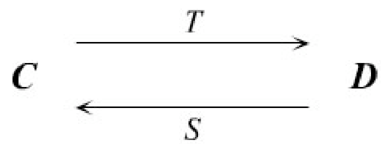
$$ST(X) \approx X \qquad \text{and} \qquad TS(Y) \approx Y$$



where v_X is an isomorphism.

We say S and T are inverses of each other up to an isomorphism.

6. Adjoint Functors



If there exist functors T and S as above and natural transformations η and ε such that

$$\eta:1_C \to ST \qquad \text{and} \qquad \varepsilon:TS \to 1_D$$

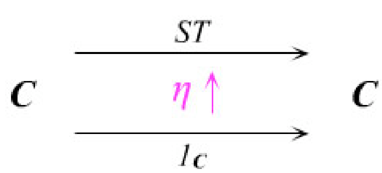
and the following conditions hold

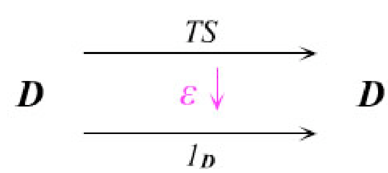
$$\eta T \cdot T\varepsilon = 1_T \qquad \text{and} \qquad S\varepsilon \cdot \eta S = 1_S$$

then T and S are **adjoint**.

These conditions will need some explanation.

$$\eta:1_C \to ST \qquad \text{and} \qquad \varepsilon:TS \to 1_D$$





have morphisms

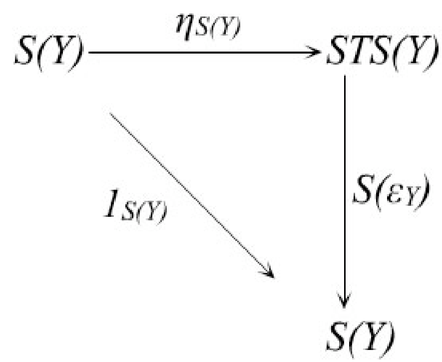
$$\eta_X:X \to ST(X) \qquad \text{and} \qquad \varepsilon_Y:TS(Y) \to Y$$

For an object S(Y) in C we have

$$\eta_{S(Y)}:S(Y) \to STS(Y)$$

and we have $\varepsilon_Y:TS(Y) \to Y$ in **D** so we are ensured an $S(\varepsilon_Y):STS(Y) \to S(Y)$ in **C**.

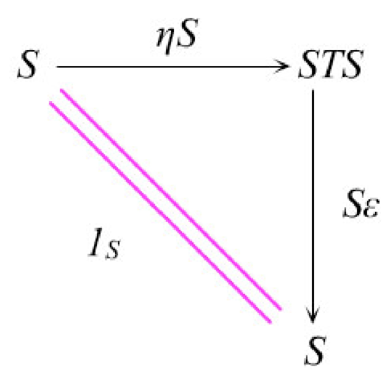
We display these two morphisms in the following diagram



The condition for adjointness is the arrow from S(Y) to S(Y) equal to the identity morphism for S(Y) to make the diagram commute.

The functor we name $\eta S:S \rightarrow STS$ associates with each Y in **D** a morphism $\eta_{S(Y)}:S(Y) \rightarrow STS(Y)$ as defined previously.

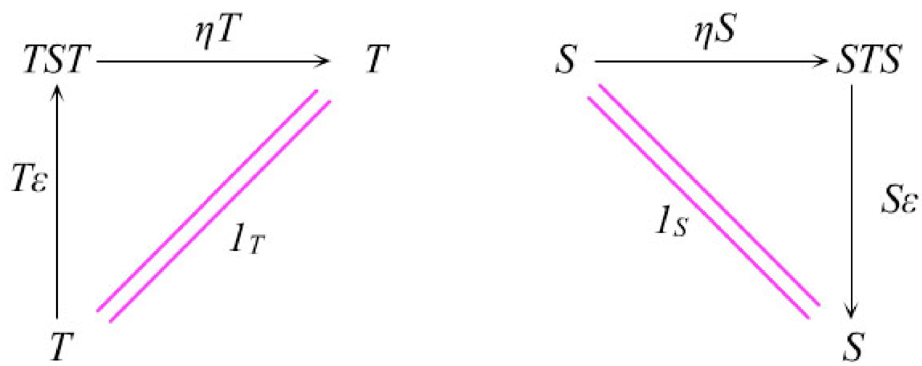
The functor we name $S\varepsilon:STS \rightarrow S$ associates with every Y in **D** a morphism $S(\varepsilon_Y):STS(Y) \rightarrow S(Y)$ as defined previously.



The composition of these two functors must equal the identity functor for S.

We have a similar triangle in **C** for the other condition. We present the two conditions thus:

$$\eta T \cdot T\varepsilon = 1_T \qquad \text{and} \qquad S\varepsilon \cdot \eta S = 1_S$$



7. An Example in Logic

We consider a category **A** = <A_0, A_1, d, c, m, e> with A_0 the class of all logical statements and morphisms A_1 being implications between them.

Existance of an f:A → B means that A logically implies B. i.e. A → B is always true.

Morphisms compose as required:
f:A → B and g:B → C then A ⇒ B and B ⇒ C so a morphism from A to C exists.

gf: A → C

since if A ⇒ B ^ B ⇒ C is always true, then so is A ⇒ C.

There can at most be one morphism between a given two objects in A. Either an implication exists uniquely or not at all.

For objects A, B, C, D and morphisms f, g, h:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

h(gf), (hg)f:A → D

h(gf) and (hg)f are equal since they share domain and codomain in A. This shows composition in A to be associative, as required.

We define two functors T, S:**A** → **A** as follows

$$T(X) = \alpha \wedge X \quad \text{and} \quad S(Y) = \alpha \Rightarrow Y$$

We want to show that T and S are adjoint. To do so we need firstly to show that the natural transformations $\eta:1_C \to ST$ and $\varepsilon:TS \to 1_D$ exist with morphisms

$$\eta_X:X \to ST(X) \qquad \text{and} \qquad \varepsilon_X:TS(X) \to X$$

Since morphisms are logical implications, the existence of these two morphisms for each X is the same as asking "Does X logically imply ST(X)?" and "Does TS(X) logically imply X?"

$S(T(X)) \equiv \alpha \Rightarrow (\alpha \wedge X)$
$\equiv \neg\alpha \vee (\alpha \wedge X)$
$\equiv (\neg\alpha \vee \alpha) \wedge (\neg\alpha \vee X)$
$\equiv \boldsymbol{T} \wedge (\neg\alpha \vee X)$
$\equiv \neg\alpha \vee X$
$\equiv \alpha \Rightarrow X$

$X \Rightarrow ST(X)$
$\equiv X \Rightarrow (\alpha \Rightarrow X)$
$\equiv \boldsymbol{T}$

Which means an implication (morphism) from X to ST(X) exists for every X in **A**.

$\eta_X: X \rightarrow ST(X)$

In answer to the second question

$T(S(X)) \equiv \alpha \wedge (\alpha \Rightarrow X)$
$\equiv \alpha \wedge (\neg\alpha \vee X)$
$\equiv (\alpha \wedge \neg\alpha) \vee (\neg\alpha \wedge X)$
$\equiv \boldsymbol{F} \vee (\neg\alpha \wedge X)$
$\equiv \neg\alpha \wedge X$

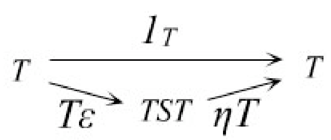
$TS(X) \Rightarrow X$
$\equiv (\neg\alpha \wedge X) \Rightarrow X$
$\equiv \boldsymbol{T}$

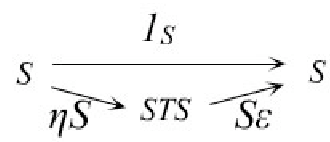
Which means an implication (morphism) from TS(X) to X exists for every X in **A**.

$\varepsilon_X: X \rightarrow TS(X)$

Further, we need to check the conditions

$$\eta T \cdot T\varepsilon = 1_T \qquad \text{and} \qquad S\varepsilon \cdot \eta S = 1_S$$





but since the morphisms of $\eta T \cdot T\varepsilon$ and 1_T share domain and codomain, they are equal for every X. The same applies to $S\varepsilon$ and $\eta S = 1_S$. Thus these conditions are satisfied easily by uniqueness of morphisms between two fixed objects in **A**.

Thus we have shown that T, conjunction, and S, implication, are adjoint in the category **A** of logical statements.

8. Conclusions

In some sense we have seen how conjunction and implication define each other; that they are not simply unrelated concepts. Other profound examples of adjoint functor pairs can be found throughout mathematics to provide insight which had not been previously achieved.

9. Bibliography

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